A NOTE ON FOURIER-LAPLACE TRANSFORM AND ANALYTIC WAVE FRONT SET IN THEORY OF TEMPERED ULTRAHYPERFUNCTIONS

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Abstract. In this paper we study the Fourier-Laplace transform of tempered ultrahyperfunctions introduced by Sebastião e Silva and Hasumi. We establish a generalization of Paley-Wiener-Schwartz theorem for this setting. This theorem is interesting in connection with the microlocal analysis. For this reason, the paper also contains a description of the singularity structure of tempered ultrahyperfunctions in terms of the concept of analytic wave front set.

1. Introduction

Tempered ultrahyperfunctions were introduced in papers of Sebastião e Silva [1, 2] and Hasumi [3], under the name of tempered ultradistributions, as the strong dual of the space of test functions of rapidly decreasing entire functions in any horizontal strip. While Sebastião e Silva [1] used extension procedures for the Fourier transform combined with holomorphic representations and considered the case of one variable, Hasumi [3] used duality arguments in order to extend the notion of tempered ultrahyperfunctions for the case of n-variables (see also [2, Section 11]). In a brief tour, Marimoto [4] gave some more precise informations concerning the work of Hasumi. More recently, the relation between the tempered ultrahyperfuntions and Schwartz distributions and some major results, as the kernel theorem and the Fourier-Laplace transform have been established by Brüning and Nagamachi in [5]. Further, aside from the mathematical interest of the results presented in Refs. [1]-[5], Brüning and Nagamachi have conjectured that the properties of tempered ultrahyperfunctions are well adapted for their use in quantum field theory with a fundamental length, while Bollini and Rocca [6] have given a general definition of convolution between two arbitrary tempered ultrahyperfunctions in order to treat the problem of singular products of functions Green also in quantum field theory. In another interesting recent work [7], Schmidt has given an insight in the operations of duality and Fourier

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transform on the space of test and generalized functions belonging to new subclasses of Fourier hyperfunctions of mixed type, satisfying polynomial growth conditions at infinity, which is very similar to the studies by Sebastião e Silva [1] and Hasumi [3] about tempered ultrahyperfunctions, and eventually suggests applications to quantum field theory.

In this article, we will give some precisions on the Fourier-Laplace transform theorem for tempered ultrahyperfunctions, by considering the theorem in its simplest form: the equivalence between support properties of a distribution in a closed convex cone and the holomorphy of its Fourier-Laplace transform in a suitable tube with conical basis. All cones will have their vertices at the origin. After some preliminares presented in Section 2, where for the sake of completeness we include a brief exposition of the basic facts concerning the theory of tempered ultrahyperfunctions, in Section 3 we define a space of functions whose elements are holomorphic in tube domains corresponding to open convex cones. In Section 4, we extend the Paley-Wiener-Schwartz (PWS) theorem for the setting of tempered ultrahyperfunctions by combining two lemmas established in Section 3. In this setting, the PWS theorem deals with the Fourier-Laplace transform of distributions of exponential growth with support in a closed convex cone. This result is also interesting in connection with the concept of analytic wave front set. For this reason, in Section 5 we study the singularity structure of tempered ultrahyperfunctions corresponding to an open cone $C \subset \mathbb{R}^n$ via the notion of analytic wave front set, a refined description of the singularity spectrum, with several applications all around Mathematics and Physics. Our aim is to provide the microlocal analysis in the space of tempered ultrahyperfunctions which is very similar to microlocal analysis in the framework of distributions.

We note that the results obtained here are of importance in the construction and study of nonstrictly localizable quantum field theories, namely, the *quasilocal* field theories (where the fields are localizable only in regions greater than a certain scale of nonlocality), and in fact they have been motivated by recent results used in the axiomatic formulation of quantum field theory with a minimum length [5]. The physical applications of the results given in this paper will appear in a coming paper, in particular, to quantum field theory in noncommutative spacetimes [8].

2. A Glance at the Theory of Tempered Ultrahyperfunctions: Definitions and Basic Properties

We shall recall in this paragraph some definitions and basic properties of the tempered ultrahyperfunction space introduced by Sebastião e Silva [1, 2] and Hasumi [3]. We shall adopt here the point of view of not entering into all technical aspects concerning the theory of tempered ultrahyperfunctions, reminding the reader to consult the Refs. [1]-[5] for more details.

Notations: We will use the standard multi-index notation. Let $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) be the real (resp. complex) n-space whose generic points are denoted by $x = (x_1, \dots, x_n)$ (resp. $z = (z_1, \dots, z_n)$), such that $x+y = (x_1+y_1, \dots, x_n+y_n)$, $\lambda x = (\lambda x_1, \dots, \lambda x_n)$, $x \geq 0$ means $x_1 \geq 0, \dots, x_n \geq 0$, $\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$ and $|x| = |x_1| + \cdots + |x_n|$. Moreover, we define $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_o^n$, where $\mathbb{N}_o$ is the set of non-negative integers, such that the length of α is the corresponding ℓ^1-norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha+\beta$ denotes $(\alpha_1+\beta_1, \dots, \alpha_n+\beta_n)$, $\alpha \geq \beta$ means $(\alpha_1 \geq \beta_1, \dots, \alpha_n \geq \beta_n)$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, and

$$D^\alpha \varphi(x) = \frac{\partial^{|\alpha|} \varphi(x_1, \dots, x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_1} \dots \partial x_n^{\alpha_n}} \ .$$

We consider two n-dimensional spaces – x-space and ξ-space – with the Fourier transform defined

$$\widehat{f}(\xi) = \mathscr{F}[f(x)](\xi) = \int_{\mathbb{R}^n} f(x) e^{i\langle \xi, x\rangle} d^n x \ ,$$

while the Fourier inversion formula is

$$f(x) = \mathscr{F}^{-1}[\widehat{f}(\xi)](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-i\langle \xi, x\rangle} d^n \xi \ .$$

The variable ξ will always be taken real while x will also be complexified – when it is complex, it will be noted $z = x + iy$.

We shall consider the function

$$h_K(\xi) = \sup_{x \in K} \bigl|\langle \xi, x\rangle\bigr| \ , \quad \xi \in \mathbb{R}^n \ ,$$

the indicator of K, where K is a compact set in $\mathbb{R}^n$. $h_K(\xi) < \infty$ for every $\xi \in \mathbb{R}^n$ since K is bounded. For sets $K = \bigl[-k, k\bigr]^n$, $0 < k < \infty$, the indicator function $h_K(\xi)$ can be easily determined:

$$h_K(\xi) = \sup_{x \in K} \bigl|\langle \xi, x\rangle\bigr| = k|\xi| \ , \quad \xi \in \mathbb{R}^n \ , \quad |\xi| = \sum_{i=1}^{n} |\xi_i| \ .$$

Let K be a convex compact subset of $\mathbb{R}^n$, then $H_b(\mathbb{R}^n; K)$ (b stands for bounded) defines the space of all functions $\in C^\infty(\mathbb{R}^n)$ such that $e^{h_K(\xi)} D^\alpha f(\xi)$ is bounded in $\mathbb{R}^n$ for any multi-index α. One defines in $H_b(\mathbb{R}^n; K)$ seminorms

$$\|\varphi\|_{K,N} = \sup_{x \in \mathbb{R}^n; \alpha \leq N} \bigl\{ e^{h_K(\xi)} |D^\alpha f(\xi)| \bigr\} < \infty \ , \quad N = 0, 1, 2, \dots \ . \tag{2.1}$$

Theorem 1. *The space $H_b(\mathbb{R}^n; K)$ equipped with the topology given by the seminorms (2.1) is a Fréchet space.*

Proof. See [3, 4]. □

If $K_1 \subset K_2$ are two compact convex sets, then $h_{K_1}(\xi) \leq h_{K_2}(\xi)$, and thus the canonical injection $H_b(\mathbb{R}^n; K_2) \hookrightarrow H_b(\mathbb{R}^n; K_1)$ is continuous. Let O be a convex open set of $\mathbb{R}^n$. To define the topology of $H(\mathbb{R}^n; O)$ it suffices to let K range over an increasing sequence

of convex compact subsets $K_1, K_2, \ldots$ contained in O such that for each $i = 1, 2, \ldots$, $K_i \subset K^\circ_{i+1}$ (K°_{i+1} denotes the interior of K_{i+1}) and $O = \bigcup_{i=1}^\infty K_i$. Then the space $H(\mathbb{R}^n; O)$ is the projective limit of the spaces $H_b(\mathbb{R}^n; K)$ according to restriction mappings above, *i.e.*

$$H(\mathbb{R}^n; O) = \lim_{K \subset O} \mathrm{proj}\ H_b(\mathbb{R}^n; K) \ , \tag{2.2}$$

where K runs through the convex compact sets contained in O.

Theorem 2. *For the spaces introduced above the following statements hold:*

1. *The space $\mathscr{D}(\mathbb{R}^n)$ of all C^∞-functions on $\mathbb{R}^n$ with compact support is dense in $H(\mathbb{R}^n; K)$ and $H(\mathbb{R}^n; O)$.*
2. *The space $H(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H(\mathbb{R}^n; O)$.*

Proof. See [3, 4]. □

From Theorem 2 we have the following injections [4]:

$$H'(\mathbb{R}^n; K) \hookrightarrow H'(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n) \ ,$$

and

$$H'(\mathbb{R}^n; O) \hookrightarrow H'(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n) \ .$$

The dual space $H'(\mathbb{R}^n; O)$ of $H(\mathbb{R}^n; O)$ is the space of distributions V of exponential growth [4] such that

$$V = D^\gamma_\xi [e^{h_K(\xi)} g(\xi)] \ ,$$

where $g(\xi)$ is a bounded continuous function.

Now, we pass to the definition of tempered ultrahyperfunctions. In the space $\mathbb{C}^n$ of n complex variables $z_i = x_i + i y_i$, $1 \le i \le n$, we denote by $T(\Omega) = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ the tubular set of all points z, such that $y_i = \operatorname{Im} z_i$ belongs to the domain Ω, *i.e.*, Ω is a connected open set in $\mathbb{R}^n$ called the basis of the tube $T(\Omega)$. Let K be a convex compact subset of $\mathbb{R}^n$, then $\mathfrak{H}_b(T(K))$ defines the space of all continuous functions φ on $T(K)$ which are holomorphic in the interior $T(K^\circ)$ of $T(K)$ such that the estimate

$$|\varphi(z)| \le C(1 + |z|)^{-N} \tag{2.3}$$

is valid for some constant $C = C_{K,N}(\varphi)$. The best possible constants in (2.3) are given by a family of seminorms in $\mathfrak{H}_b(T(K))$

$$\|\varphi\|_{K,N} = \sup_{z \in T(K)} \left\{(1 + |z|)^N |\varphi(z)|\right\} < \infty \ , \quad N = 0, 1, 2, \ldots \ . \tag{2.4}$$

Theorem 3. *The space $\mathfrak{H}_b(T(K))$ equipped with the topology given by the seminorms (2.4) is a Fréchet space.*

Proof. See [3, 4]. □

The fact of the spaces $\mathfrak{H}_b(T(K))$ belong to the class of nuclear Fréchet spaces is important for applications to QFT.

If $K_1 \subset K_2$ are two convex compact sets, then $\mathfrak{H}_b(T(K_2)) \hookrightarrow \mathfrak{H}_b(T(K_1)$. Given that the spaces $\mathfrak{H}_b(T(K_i))$ are Fréchet spaces, the space $\mathfrak{H}(T(O))$ is characterized as a projective limit of Fréchet spaces

$$\mathfrak{H}(T(O)) = \lim_{K\subset O} \operatorname{proj} \mathfrak{H}_b(T(K)) \ , \tag{2.5}$$

where K runs through the convex compact sets contained in O and the projective limit is taken following the restriction mappings above.

Proposition 1. *If $f \in H(\mathbb{R}^n; O)$, the Fourier transform of f belongs to the space $\mathfrak{H}(T(O))$, for any open convex nonempty set $O \subset \mathbb{R}^n$. By the dual Fourier transform $H'(\mathbb{R}^n; O)$ is topologically isomorphic with the space $\mathfrak{H}'(T(-O))$.*

Proof. See [4]. □

Definition 1. *A* **tempered ultrahyperfunction** *is a continuous linear functional defined on the space of test functions $\mathfrak{H} = \mathfrak{H}(T(\mathbb{R}^n))$ of rapidly decreasing entire functions in any horizontal strip. The space of all tempered ultrahyperfunctions is denoted by $\mathscr{U}(\mathbb{R}^n)$.*

The space $\mathscr{U}(\mathbb{R}^n)$ is characterized in the following way [3]; let $\mathscr{H}_{\boldsymbol{\omega}}$ be the space of all functions $f(z)$ such that:

- $f(z)$ is analytic for $\{z \in \mathbb{C}^n \mid |\operatorname{Im} z_1| > p, |\operatorname{Im} z_2| > p, \ldots, |\operatorname{Im} z_n| > p\}$.
- $f(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n \mid |\operatorname{Im} z_1| \geqq p, |\operatorname{Im} z_2| \geqq p, \ldots, |\operatorname{Im} z_n| \geqq p\}$, where $p = 0, 1, 2, \ldots$ depends on $f(z)$.
- $f(z)$ is bounded by a power of z: $|f(z)| \leq C(1 + |z|)^N$, where C and N depend on $f(z)$.

Let $\mathbf{\Pi}$ be the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$. Then $\mathscr{U}$ is the quotient space $\mathscr{U} = \mathscr{H}_{\boldsymbol{\omega}}/\mathbf{\Pi}$. By a pseudo-polynomial we understand a function of z of the form $\sum_s z_j^s G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n)$, with $G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \mathscr{H}_{\boldsymbol{\omega}}$.

According to Hasumi [3, Prop.5] the dual $\mathfrak{H}'$ of $\mathfrak{H}$ is algebraically isomorphic with the space $\mathscr{U}$.

3. Tempered Ultrahyperfunctions Corresponding to a Cone: The Space $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$

Let us introduce for the beginning some terminology and simple facts concerning cones. An open set $C \subset \mathbb{R}^n$ is called a cone if $x \in C$ implies $\lambda x \in C$ for all $\lambda > 0$. Moreover, C is an open connected cone if C is a cone and if C is an open connected set. In the sequel, it will be sufficient to assume for our purposes that the open connected cone C in $\mathbb{R}^n$ is an open convex cone with vertex at the origin. A cone C' is called compact in C – we write

$C' \Subset C$ – if the projection $\mathsf{pr}\overline{C}' \stackrel{\text{def}}{=} \overline{C}' \cap S^{n-1} \subset \mathsf{pr}C \stackrel{\text{def}}{=} C \cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in x-space, we associate with C a closed convex cone C^* in ξ-space which is the set $C^* = \big\{\xi \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq 0, \forall x \in C\big\}$. The cone C^* is called the *dual cone* of C (see Fig. 1).

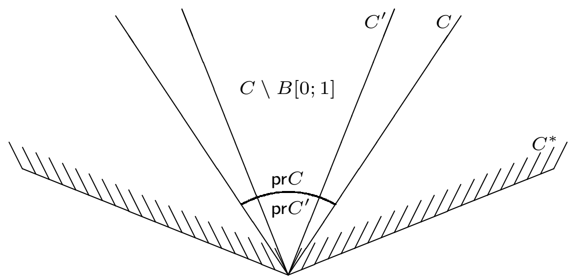

Figure 1

By $T(C)$ we will denote the set $\mathbb{R}^n + iC \subset \mathbb{C}^n$. If C is open and connected, $T(C)$ is called the tubular radial domain in $\mathbb{C}^n$, while if C is only open $T(C)$ is referred to as a tubular cone. An important example of tubular radial domain in quantum field theory is the forward light-cone

$$V_+ = \Big\{z \in \mathbb{C}^n \mid \operatorname{Im} z_1 > \Big(\sum_{i=2}^{n} \operatorname{Im}^2 z_i\Big)^{\frac{1}{2}}\Big\} .$$

We will deal with tubes defined as the set of all points $z \in \mathbb{C}^n$ such that

$$T(C) = \Big\{x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C, |y| < \delta\Big\} ,$$

where $\delta > 0$ is an arbitrary number.

Let C be an open convex cone and let C' be an arbitrary compact cone of C. Let $B[0;r]$ denote a closed ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C';r) = \mathbb{R}^n + iC' \setminus \big(C' \cap B[0;r]\big)$. We are going to introduce a space of holomorphic functions which satisfy certain estimate according to Carmichael and Milton [9]. We want to consider the space consisting of holomorphic functions $f(z)$ such that

$$\big|f(z)\big| \leq K(C')(1 + |z|)^N e^{h_{C^*}(y)} , \quad z = x + iy \in T(C';r) , \tag{3.1}$$

where $h_{C^*}(y) = \sup_{\xi \in C^*} |\langle \xi, y\rangle|$ is the indicator of C^*, $K(C')$ is a constant that depends on an arbitrary compact cone C' and N is a non-negative real number. The set of all functions $f(z)$ which are holomorphic in $T(C';r)$ and satisfy the estimate (3.1) will be denoted by $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$. In what follows, we shall prove two lemmas which will be important for our extension of PWS theorem for the setting of tempered ultrahyperfunctions. The proofs of lemmas are slight variations of that of Lemma 10 and Lemma 11 in [9]. Throughout the remainder of this paper $T(C';r)$ will denote the set $\mathbb{R}^n + iC' \setminus \big(C' \cap B[0;r]\big)$.

Lemma 1. *Let C be an open convex cone, and let C' be an arbitrary compact cone contained in C. Let $h(\xi) = e^{k|\xi|}g(\xi)$, $\xi \in \mathbb{R}^n$, be a function with support in C^*, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$. Let y be an arbitrary but fixed point of $C' \setminus \big(C' \cap B[0;r]\big)$. Then $e^{-\langle \xi,y\rangle}h(\xi) \in L^2$, as a function of $\xi \in \mathbb{R}^n$.*

Proof. By Vladimirov [11, Lemma 2, p.223] there is a real number $1 \geq c = c(C') > 0$ such that $\langle \xi, y\rangle \geq c|\xi||y|$ for every $\xi \in C^*$ and $y \in C'$. Then, by using the fact that $\sup_{\xi\in C^*}|g(\xi)| \leq M$, it follows that

$$\left|e^{-\langle \xi,y\rangle}h(\xi)\right| \leq Me^{k|\xi|-c|\xi||y|} \ . \tag{3.2}$$

From (3.2) we have that

$$\int_{\mathbb{R}^n}\left|e^{-\langle \xi,y\rangle}h(\xi)\right|^2 d\xi = \int_{C^*}\left|e^{-\langle \xi,y\rangle}h(\xi)\right|^2 d\xi \leq M^2\int_{C^*} e^{-2(c|\xi||y|-k|\xi|)}d\xi \ . \tag{3.3}$$

Using a result concerning the Lesbegue integral (see Schwartz [12, Prop.32, p.39]) and the assumption that $k < c|y|$ for fixed k, we get

$$\int_{\mathbb{R}^n}\left|e^{-\langle \xi,y\rangle}h(\xi)\right|^2 d\xi \leq M^2S^{n-1}\int_0^\infty e^{-2(c|y|-k)t}t^{n-1}dt \ , \tag{3.4}$$

where S^{n-1} is the area of the unit sphere in $\mathbb{R}^n$. Integrating by parts $(n-1)$ times on the last integral in (3.4), it follows that

$$\int_{\mathbb{R}^n}\left|e^{-\langle \xi,y\rangle}h(\xi)\right|^2 d\xi \leq M^2S^{n-1}(n-1)!(2c|y|-2k)^{-n} \ . \tag{3.5}$$

with y fixed in $C' \setminus \big(C' \cap B[0;r]\big)$. Thus the r.h.s. of (3.5) is finite. This implies that $e^{-\langle \xi,y\rangle}h(\xi) \in L^2$, as a function of $\xi \in \mathbb{R}^n$, for y fixed in $C' \setminus \big(C' \cap B[0;r]\big)$. □

Definition 2. *We denote by $H'_{C^*}(\mathbb{R}^n;O)$ the subspace of $H'(\mathbb{R}^n;O)$ of distributions of exponential growth with support in the cone C^*:*

$$H'_{C^*}(\mathbb{R}^n;O) = \Big\{V \in H'(\mathbb{R}^n;O) \mid \operatorname{supp}(V) \subseteq C^*\Big\} \ . \tag{3.6}$$

Lemma 2. *Let C be an open convex cone, and let C' be an arbitrary compact cone contained in C. Let $V = D^\gamma_\xi[e^{h_K(\xi)}g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k,k\big]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n;O)$. Then $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi,z\rangle})$ is an element of $\mathscr{H}^{\boldsymbol{o}}_{\boldsymbol{c}}$.*

Proof. The proof that $\operatorname{supp}(V) \subseteq C^*$ implies that $f(z)$ is holomorphic in $T(C';r)$ is obtained by considering formula:

$$\begin{aligned} f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi,z\rangle}) &= (2\pi)^{-n}\int_{C^*} D^\gamma_\xi[e^{k|\xi|}g(\xi)]e^{-i\langle \xi,z\rangle}d^n\xi \\ &= (2\pi)^{-n}(-i)^{|\gamma|}z^\gamma\int_{C^*}[e^{k|\xi|}g(\xi)]e^{-i\langle \xi,z\rangle}d^n\xi \ . \end{aligned} \tag{3.7}$$

In order to prove that $f(z)$ is holomorphic, it is enough to consider the function

$$h(z) = \int_{C^*} [e^{k|\xi|} g(\xi)] e^{-i\langle \xi, z\rangle} d^n\xi \ . \tag{3.8}$$

Let z_o be an arbitrary but fixed point of $T(C';r)$ and let $R(z_o;a) \subset T(C';r)$ be an arbitrary but fixed neighborhood of z_o with radius a, such that its closure is in $T(C';r)$. Since $R(z_o;a)$ is fixed and has closure in $T(C';r)$, we can find two balls of the origin in $\mathbb{R}^n$ of radius k and δ, respectively, so that $0 < r < k < |y| < \delta$ for all $y = \mathrm{Im}(z)$, with $z = x + iy \in R(z_o;a)$ (see Fig. 2).

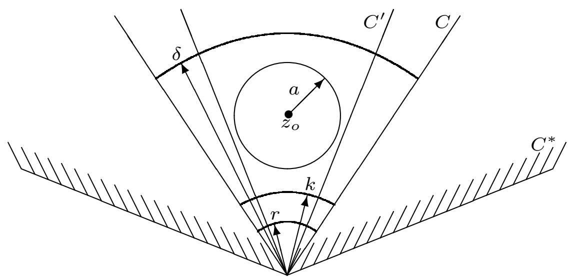

Figure 2

Taking the absolute value of both sides of (3.8) and using the fact that $g(\xi)$ is bounded, we conclude that

$$\begin{aligned} |h(z)| = \Big|\int_{C^*} [e^{k|\xi|} g(\xi)] e^{-i\langle \xi, z\rangle} d\xi\Big| &\leq \int_{C^*} |g(\xi)| e^{k|\xi| + \langle \xi, y\rangle} d^n\xi \\ &\leq \sup_{\xi\in C^*} |g(\xi)| \int_{C^*} e^{k|\xi| + \langle \xi, y\rangle} d^n\xi \\ &\leq M \int_{C^*} e^{k|\xi| + \langle \xi, y\rangle} d^n\xi \ . \end{aligned} \tag{3.9}$$

Choose an arbitrary but fixed $Y \in C'$ such that $z = x + iY \in R(z_o;a)$. Assume that ξ belongs to the open half-space $\big\{\xi \in C^* \mid \langle \xi, Y\rangle < 0\big\}$. Then, for some fixed number $c(Y) > 0$, it follows that $\langle \xi, Y\rangle \leq -c(Y)|\xi|$ for $\xi \in C^*$. Thus, with the assumption that $k < c(Y)$ for fixed k, we repeat part of the argument used in proof of Lemma 1, namely, we use the result in Schwartz [12, Prop.32, p.39] concerning the Lesbegue integral to get

$$|h(z)| \leq M S^{n-1} \int_0^\infty e^{-(c(Y)-k)t} t^{n-1} dt = M S^{n-1} (n-1)! (c(Y) - k)^{-n} \ , \tag{3.10}$$

where S^{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Now, by differentiation of (3.8), we immediately obtain that

$$|D_z^\beta h(z)| \leq M S^{n-1} \int_0^\infty e^{-(c(Y)-k)t} t^{|\beta|+n-1} dt = M S^{n-1} (n-1)! (c(Y) - k)^{-(|\beta|+n)} \ . \tag{3.11}$$

This shows that the integral defining $h(z)$ and any complex derivative, $D_z^{\beta}h(z)$, converges uniformly for $z \in R(z_o; a)$. Since z_o is an arbitrary point in $T(C'; r)$, it follows that $h(z)$ exists and is holomorphic for $z \in T(C'; r)$. In turn, this implies that $f(z)$ exists and is holomorphic for $z \in T(C'; r)$. From (3.7) and (3.10) it follows that the existence of a constant $K(C')$ and a positive real number N implies that

$$|f(z)| \leq (2\pi)^{-n}|z^{\gamma}||h(z)| \leq K(C')(1+|z|)^N \qquad z = x+iy \in T(C'; r)\ .$$

Now, since for $y \in C' \setminus \big(C' \cap B[0; r]\big)$, $\sup_{\xi \in C^*} e^{|\langle \xi, y\rangle|} > 1$, then

$$|f(z)| \leq K(C')(1+|z|)^N \sup_{\xi \in C^*} e^{|\langle \xi, y\rangle|} = K(C')(1+|z|)^N e^{h_{c^*}(y)}\ ,$$

for $z = x+iy \in T(C'; r)$, from which follows the lemma. □

Remark 1. A result as the Lemma 2 was obtained by Carmichael and Milton [9] and Pathak [14] to other spaces of distributions. In [9] Carmichael and Milton proved a result of this type for the dual spaces of the spaces of type $\mathscr{S}$ introduced by Gel'fand and Shilov [13]. Using techniques as in the paper of Carmichael and Milton [9], Pathak [14] proved similar result for tempered ultradistributions, based on classes of ultradifferentiable functions.

4. A Generalization of the Paley-Wiener-Schwartz Theorem

In what follows, we shall show that more can be said concerning the functions $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$. It will be shown that $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ can be recovered as the (inverse) Fourier-Laplace transform* of the constructed distribution $V \in H'_{C^*}(\mathbb{R}^n; O)$. This result is a generalization of the PWS theorem.

Theorem 4 (Paley-Wiener-Schwartz-type Theorem). *Let $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$, where C is an open convex cone. Then the distribution $V \in H'_{C^*}(\mathbb{R}^n; O)$ has a uniquely determined inverse Fourier-Laplace transform $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi, z\rangle})$ which is holomorphic in $T(C'; r)$ and satisfies the estimate (3.1).*

Proof. Consider

$$h_y(\xi) = \int_{\mathbb{R}^n} \frac{f(z)}{P(iz)}\, e^{i\langle \xi, z\rangle} d^n x\ , \quad z \in T(C'; r)\ , \tag{4.1}$$

with $h_y(\xi) = e^{k|\xi|} g_y(\xi)$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$, and $P(iz) = (-i)^{|\gamma|} z^{\gamma}$. By hypothesis $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ and satisfies (3.1). For this reason, for an n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n)$ of non-negative integers conveniently chosen, we obtain

$$\Big|\frac{f(z)}{P(iz)}\Big| \leq K(C')(1+|z|)^{-n-\varepsilon} e^{h_{c^*}(y)}\ , \tag{4.2}$$

*The convention of signs in the Fourier transform which is used here one leads us to consider the inverse Fourier-Laplace transform.

where n is the dimension and ε is any fixed positive real number. This implies that the function $h_y(\xi)$ exists and is a continuous function of ξ. Further, by using arguments paralleling the analysis in [11, p.225] and the Cauchy-Poincaré Theorem [11, p.198], we can show that the function $h_y(\xi)$ is independent of $y = \operatorname{Im} z$. Therefore, we denote the function $h_y(\xi)$ by $h(\xi)$.

From (4.2) we have that $f(z)/P(iz) \in L^2$ as a function of $x = \operatorname{Re} z \in \mathbb{R}^n$, $y \in C' \setminus \big(C' \cap B[0;r]\big)$. Hence, from (4.1) and the Plancherel theorem we have that $e^{-\langle \xi, y\rangle} h(\xi) \in L^2$ as a function of $\xi \in \mathbb{R}^n$, and

$$\frac{f(z)}{P(iz)} = \mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} h(\xi)\big](x)\ , \quad z \in T(C';r)\ , \tag{4.3}$$

where the inverse Fourier transform is in the L^2 sense. Here, Parseval's equation holds:

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \Big|e^{-\langle \xi, y\rangle} h(\xi)\Big|^2 d^n\xi = \int_{\mathbb{R}^n} \Big|\frac{f(z)}{P(iz)}\Big|^2 d^n x\ . \tag{4.4}$$

It should be noted that for Eq.(4.3) to be true ξ must belong to the open half-space $\big\{\xi \in C^* \mid \langle \xi, y\rangle < 0\big\}$, for $y \in C' \setminus \big(C' \cap B[0;r]\big)$, as stated by Lemma 2, since by hypothesis $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.

Now, if $h(\xi) \in H'_{C^*}(\mathbb{R}^n;O)$, then $V = D_\xi^\gamma h(\xi) \in H'_{C^*}(\mathbb{R}^n;O)$. Since C^* is a regular set [12, pp.98, 99], thus $\operatorname{supp}(h) = \operatorname{supp}(V)$. By Lemma 2 $(V, e^{-i\langle \xi, z\rangle})$ exists as a holomorphic function of $z \in T(C';r)$ and satisfies the estimate (3.1). A simple calculation yields

$$(2\pi)^{-n}(V, e^{-i\langle \xi, z\rangle}) = P(iz)\mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} h(\xi)\big](x) \quad z \in T(C';r)\ . \tag{4.5}$$

In view of Lemma 1, the inverse Fourier transform can be interpreted in L^2 sense. Combining (4.3) and (4.5), we have $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi, z\rangle})$. The uniqueness follows from the isomorphism of the dual Fourier transform, according to Proposition 1. This completes the proof of the theorem. □

The following corollary is immediate from Theorem 4 and preceding construction:

Corollary 1. *Let C^* be a closed convex cone and K a convex compact set in $\mathbb{R}^n$. Define an indicator function $h_{K,C^*}(y)$, $y \in \mathbb{R}^n$, and an open convex cone C_K such that $h_{K,C^*}(y) = \sup_{\xi\in C^*}\big|H_K(\xi) - \langle \xi, y\rangle\big|$ and $C_K = \big\{y \in \mathbb{R}^n \mid h_{K,C^*}(y) < \infty\big\}$. Then the distribution $V \in H'_{C^*}(\mathbb{R}^n;O)$ has a uniquely determined inverse Fourier-Laplace transform $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi, z\rangle})$ which is holomorphic in the tube $T(C'_K;r) = \mathbb{R}^n + iC'_K \setminus \big(C'_K \cap B[0;r]\big)$, and satisfies the following estimate, for a suitable $K \subset O$,*

$$\big|f(z)\big| \leq K(C')(1 + |z|)^N e^{h_{K,C^*}(y)}\ , \tag{4.6}$$

where $C'_K \Subset C_K$.

Remark 2. A result of this type has been established by Brüning and Nagamachi [5, Thm.2.15]. The space of holomorphic functions $f(z)$ considered by Brüning and Nagamachi restricted to C_K is a subspace of the space $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ defined in this paper. While the function $f(z)$ considered by Brüning and Nagamachi satisfies the growth condition (4.6) and is holomorphic in the interior of $\mathbb{R}^n \times iC_K$, in our case $f(z)$ satisfies (4.6) but is required to be holomorphic in $\mathbb{R}^n \times iC'_K \setminus \big(C'_K \cap B[0;r]\big)$ only.

5. Analytic Wave Front Set of Tempered Ultrapyperfunctions

This section is about the singularity structure of tempered ultrahyperfunctions. Here, we shall follow the results and ideas of [10] and characterize the singularities of tempered ultrahyperfunctions via the notion of analytic wave front set. Define $\mathscr{U}_c = \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}/\boldsymbol{\Pi}$ as being the quotient space of $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ by set of pseudo-polynomials. The set $\mathscr{U}_c$ is the space of tempered ultrahyperfunctions corresponding to the open cone $C \subset \mathbb{R}^n$. Let us now consider the consequences of Theorem 4.

Theorem 5. *If $u \in \mathscr{U}_c(\mathbb{R}^n)$ and $V \in H'_{C^*}(\mathbb{R}^n;O)$, then $WF_A(u) \subset \mathbb{R}^n \times C^*$.*

Indication of proof. Taking into account that tempered ultrahyperfunctions are representable by means of holomorphic functions, we use the integral representation of such objects according to Proposition 11.1 in [1] (which has a similar characterization for the case n-dimensional). Thus, according to Proposition 11.1 in [1], every element $u \in \mathscr{U}_c(\mathbb{R}^n)$ is representable under the form

$$u = \int_{\mathbb{R}^n} V(\xi) e^{-i\langle \xi, z\rangle} d^n\xi = \int_{C^*} V(\xi) e^{-i\langle \xi, z\rangle} d^n\xi = f(z) \ , \tag{5.1}$$

where V is a distribution of exponential type. Hence, we can determine the $WF_A(u)$ by just looking at the behavior of $f(z)$, where $f(z)$ is any representative of an element $u \in \mathscr{U}_c(\mathbb{R}^n)$. By Paley-Wiener-Schawartz-type theorem, Theorem 4, $f(z)$ is holomorphic at $T(C';r)$ unless $\langle \xi, Y\rangle \geq 0$ for $\xi \in C^*$ and $Y \in C'$, with $|Y| < \delta$. Since Y has an arbitrary direction in C', this shows that

$$WF_A(u) \subset \mathbb{R}^n \times \big\{\xi \in \mathbb{R}^n \setminus \{0\} \mid \langle \xi, Y\rangle \geq 0\big\} \ ,$$

which is the desired result. □

Note Added: After the text of the present paper was submitted for publication, we learn that Carmichael already has published an article which contains some similar results related to the our construction, especially to the Sections 3 and 4: R.D. Carmichael, "*The tempered ultra-distributions of J. Sebastião e Silva,*" **Portugaliae Mathematica 36** (1977) 119. However, it should be noted that the singularity structure of tempered ultrahyperfunctions, here characterized by the analytic wave front set, has not been considered by Carmichael.

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